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The Sachs-Wolfe Effect

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Abstract. We present a pedagogical derivation of the Sachs-Wolfe effect, specifically the factor $\frac{1}{3}$ relating the temperature fluctuations to gravitational potentials. The result arises from a cancellation between gravitational redshifts and intrinsic temperature fluctuations which can be derived from a coordinate transformation of the background.

Key words: cosmic microwave background – Cosmology:theory

1. Introduction

On large scales, cosmic microwave background (CMB) anisotropies are related to density fluctuations by the Sachs-Wolfe (1967) effect. The gravitational effects of density perturbations on the potential Φ generates temperature fluctuations

$$\frac{\Delta T}{T} = -\frac{1}{3}\Phi \,, \tag{1}$$

in the simplest case of adiabatic fluctuations in a matter dominated universe.

In this note we present an intuitive, but mathematically rigorous, derivation of this formula and its generalizations. While the effect is well known, a simple but rigorous discussion does not appear to have been presented before. The factor $\frac{1}{3}$ is easily derived from relativistic perturbation theory (see e.g. Mukhanov, Feldman & Brandenberger 1992, Liddle & Lyth 1993, Stebbins 1993, White, Scott & Silk 1994, Stoeger, Ellis & Xu 1994, Hu 1995), but that requires quite a high level of sophistication on the part of the reader. However these derivations exemplify the use of coordinate transformations (Bardeen 1980, Kodama & Sasaki 1984) to expose the underlying physics. Given the importance of the result for normalizing large-scale structure models to the *COBE* DMR measurement of large-angle anisotropies, we feel a pedagogical introduction, using these ideas, is worthwhile.

2. Derivation

Following Sachs and Wolfe (1967), we start from the geodesic equation for photons propagating in a metric perturbed by a gravitational potential Φ. The resulting frequency shifts for the CMB photons lead to a temperature perturbation

$$\left.\frac{\Delta T}{T}\right|_f = \left.\frac{\Delta T}{T}\right|_i - \Phi_i \,, \tag{2}$$

where i and f refer to "initial" and "final" states. We have dropped the term due to the local gravitational potential (Φ_f) which gives an isotropic temperature shift. We also neglect the Doppler shift from the relative motion of the emitter and receiver and other small scale effects, and assume that the potentials are constant on large scales. We return to these assumptions later.

The interpretation of Eq. (2) is straightforward. The first term on the right-hand side is the "intrinsic" temperature perturbation at early times. The second term indicates the energy lost when the photon climbs out of a potential well. In this limit the Sachs-Wolfe effect is simply an expression of energy conservation.

In order to rederive Eq. (1), we consider the case of adiabatic fluctuations in a critical density, matter dominated universe. For adiabatic fluctuations an overdensity, or potential well, represents a larger than average number of photons or an intrinsic hot spot. We thus expect the two terms in Eq. (2) to partially cancel (see e.g. Stebbins 1993). By comparison with Eq. (1), we expect $\Delta T/T|_i = \frac{2}{3}\Phi$.

The derivation proceeds by moving to the rest frame of the cosmological fluid (photons, baryons, dark matter . . .) which is known as the "comoving" or "velocity orthogonal isotropic" gauge (Bardeen 1980, Kodama & Sasaki 1984). Here density fluctuations vanish and proper time coincides with coordinate time at large scales. The intrinsic term is negligible, which follows from the Poisson equation: $\nabla^2\Phi = 4\pi G\,\delta\rho$ or $k^2\Phi = 4\pi G\,a^2\,\delta\rho$ as $k \to 0$. Here $a(t)$ is the scale factor. That proper time and coordinate time coincide follows from the fact that we are in the rest frame of the fluid. It is also in this frame that computation of fluctuations from inflation takes on its simplest form (Mukhanov et al. 1992, Liddle & Lyth 1993).

However we wish to work in a frame where our Newtonian intuition makes sense, the so called Newtonian gauge. To get from our rest frame to the Newtonian frame requires us to perform a shift of time coordinate. (The spatial metric is isotropic in both the Newtonian frame and the rest frame which does not allow us to make a redefinition of the space coordinates.) Recall that in a gravitational potential clocks run slow

$$ds = \sqrt{1-2\Phi}\;dt \simeq (1-\Phi)dt \,. \tag{3}$$

Since the background temperature is redshifting as $aT =$ constant, in making a shift of time coordinate $t \to t+dt$ (known

also as a gauge transformation see Bardeen 1980 and Kodama & Sasaki 1984), we induce a temperature fluctuation. If the equation of state is $p = w\rho$, then $a \sim t^{2/3(1+w)}$ and

$$\left.\frac{\Delta T}{T}\right|_i = -\frac{\delta a}{a} = \frac{2}{3(1+w)}\Phi\,, \tag{4}$$

where we have used Eq. (3) in the last step. For a matter dominated universe $w = 0$, using Eq. (2) gives the factor of $\frac{1}{3}$ in Eq. (1). More generally,

$$\left.\frac{\Delta T}{T}\right|_f = -\frac{1+3w}{3+3w}\Phi\,. \tag{5}$$

There is another way to look at these two terms. If the former discussion is the "fluid" picture, this is the "metric" picture. In this picture, the second term in Eq. (2) comes from the time-time part of the metric. Time dilation changes the frequency and hence the energy of oscillators (see e.g. Weinberg 1972). By contrast, the intrinsic term is generated when we change from the rest frame to the Newtonian frame. In so doing we change the definition of the spatial hypersurfaces and thus the "volume element" or space-space part of the metric $\propto a^2$. This induces a redshift just as the normal expansion of the universe induces a redshift $\sim a^{-1}(t)$. The change in the spatial curvature in changing frames reproduces Eq. (4).

3. Isocurvature

Note that our argument works for isocurvature fluctuations also. In this case, there are no initial curvature or metric perturbations, so the frames coincide initially. There are thus no initial perturbations in the temperature at large scales. Since the potentials do not vanish today, our assumption of constant potentials needs to be relaxed. This adds a third term to Eq. (2): if the potential changes while the photon is crossing it, a net temperature shift remains between the infall blueshift and the outclimb redshift. A potential stretches space so the accompanying changes in the length scale, and hence wavelength, doubles the effect. The extra term is thus $-2\int\dot{\Phi}$. Integrating from some early time until the present, one obtains $\Delta T/T = -2\Phi$. As an aside, in the metric picture this term arises since the Lagrangian $\mathcal{L} \propto g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ depends on time. Since t is not a cyclic coordinate, energy is not conserved along the photon trajectory. The redshift goes as $a(t)$ plus the "extra" time dependence from $\Phi(t)$ (e.g. White et al. 1994).

4. Conclusions

We have presented a pedagogical derivation of the coefficient which relates the large-angle CMB temperature fluctuations to the gravitational potential. For adiabatic fluctuations, this comes about by a partial cancellation of two terms – the intrinsic temperature perturbation and the gravitational redshift from climbing out of a potential. The latter wins, meaning photon overdensities are CMB cold spots. This cancellation, which cannot be present in models with isocurvature initial conditions, is crucial to the success of the inflationary cold dark matter model in predicting small CMB fluctuations for a given amount of large-scale structure (e.g. White & Scott 1996).

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